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On Proximinality and Sets of Operators. III. Approximation by Finite Rank Operators on Spaces of Continuous Functions

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INTRODUCTION

If A is a closed subset of the normed linear space X, then A is said to be "proximinal" in X if, for each $x \in X$, there is $y_0 \in A$ such that

$$\| x - y_0 \| = d(x, A) = \inf\{ \| x - y \| ; y \in A \}.$$

In this case y_0 is called a "best approximation" for x from A. If B is a subset of X, then

$$\delta(B, A) = \sup\{d(x, A); x \in B\},$$

is the deviation of B from A, and

$$d_n(B, X) = \inf\{\delta(B, N); N \text{ is an } n\text{-dimensional subspace of } X\}$$

is the Kolmogrov n-width of B in X.

If X and Y are normed linear spaces, then $L(X, Y)$ denotes the set of all bounded linear operators from X to Y, $K(X, Y)$ the set of all compact operators in $L(X, Y)$ and $K_n(X, Y)$ the set of all operators in $L(X, Y)$ of rank $\leqslant n$.

The first serous study of the proximinality of $K_n(X, Y)$ in $K(X, Y)$ and $L(X, Y)$ appeared in the paper of Deutsch, Mach, and Saatkamp [2]. This paper was followed by two others, Kamal [4] and Kamal [5], in which several results concerning the proximinality of of $K_n(X, Y)$ in $K(X, Y)$ and $L(X, Y)$ were proved. In their paper [2], Deutsch *et al.* proved that for each integer $n \geqslant 0$, the set $K_n(c_0, c_0)$ is proximinal in $L(c_0, c_0)$, while in the present paper the following result is proved: Let Q and S be locally com-

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pact Hausdorff spaces, and assume that S contains an infinite convergent sequence of distinct elements. Then, for each $n \geqslant 1$, $K_n(C_0(Q), C_0(S))$ is proximinal in $K(C_0(Q), C_0(S))$ if and only if Q is finite.

Deutsch *et al.* asked whether or not it is true that the set $K_n(c, c_0)$ is proximinal in $L(c, c_0)$. In this paper the author continues the study of the proximinality of the set $K_n(K, Y)$ in $K(X, Y)$ and $L(X, Y)$.

In Section 1, it is shown that for each positive integer $n \geqslant 1$, the set $K_n(c, c_0)$ is not proximinal in $L(c, c_0)$, this gives a negative solution to part (a) of Problem 5.2.2 of Deutsch *et al.* [2]. In Section 2, it is shown that if $E = c$ or c_0 then for each positive integer $n \geqslant 1$, the set $K_n(E, c)$ is not proximinal in $K(E, c)$. Since by Mach [6], the set $K(c_0, c)$ is proximinal in $L(c_0, c)$, it follows that there are Banach spaces X and Y, such that the set $K_n(X, Y)$ is not proximinal in $K(X, Y)$, whereas the set $K(X, Y)$ is proximinal in $L(X, Y)$. The results of Sections 1 and 2 will be used in Section 3 to obtain the main result of this paper which can be stated as follows: if Q and S are two locally compact Hausdorff spaces, such that S contains an infinite convergent sequence of distinct elements, then for each positive integer $n \geqslant 1$, the set $K_n(C_0(Q), C_0(S))$ is proximinal in $K(C_0(Q), C_0(S))$ if and only if Q is finite. This result is not generally true if S fails to contain an infinite convergent sequence of distinct elements, Deutsch *et al.* [2] proved that for any normed linear space X, the set $K_n(X, c_0)$ is proximinal in $K(X, c_0)$. The set that contains an infinite convergent sequence of distinct elements was introduced in Kamal [5] and it was called a set that "*Contains* Q_0." It is shown also in Section 3 that if the locally compact Hausdorff space Q contains Q_0, then for each positive integer $n \geqslant 1$, the set $K_n(C_0(Q), c_0)$ is not proximinal in $L(C_0(Q), c_0)$. This might help in finding a general solution to part (b) of Problem 5.2.2 of Deutsch *et al.* [2]. The rest of the Introduction will be used to cover the basic definitions and notations that will be used later in this paper.

If Q is a Hausdorff topological space, X is a normed linear space and τ is a topology defined on X, then $C(Q, (X, \tau))$ denotes the set of all bounded function from Q to X, which are continuous with respect to τ. If $\tau = \|\cdot\|$ then $C_0(Q, X) = \{f \in C(Q, (X, \|\cdot\|)); \forall \varepsilon > 0$ the set $\{q \in Q;\ \|f(q)\| \geqslant \varepsilon\}$ is compact$\}$. If $X = R$, the set of real numbers, then $C_0(Q, R)$ is denoted by $C_0(Q)$. If Q is the set of all positive integers, then $C_0(Q, X)$ consists of all bounded sequences in X that converges to zero, and will be denoted by $c_0(X)$. If Q is the one point compactification of the set of positive integers, then $C_0(Q, X)$ consists of all bounded convergent sequences in X, and will be denoted by $c(X)$. If X^* is the dual of X, then

$$C_0(Q,(X^*, \omega^*)) = \{f \in C(Q, (X^*, \omega^*)); \hat{x} \circ f \in C_0(Q)\ \forall x \in X\},$$

where $\hat{x}$ is the image of x under the canonical injection of X in X^{**}.

The proof of the following lemma can be found in Kamal [4].

0.1. LEMMA. *Let X be a Banach space, Q a locally compact Hausdorff space, and for each nonnegative integer n, let*

$$C_n = \{f \in C_0(Q, X^*); f(Q) \subseteq N \text{ for some } n\text{-dimensional subspace } N \text{ of } X^*\}.$$

Then $K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$ [resp. $K(X, C_0(Q))$] *if and only if C_n is proximinal in $C_0(Q, (X^*, \omega^*))$* [resp. $C_0(Q, X^*)$]

0.2. DEFINITION. *Let X be a Banach space, Q a locally compact Hausdorff space, and C_n be as in Lemma* 0.1.

(a) *For each $f \in C_0(Q, (X^*, \omega^*))$, let $a_n(f)$ denotes $d(f, C_n)$.*

(b) *For each $T \in L(X, C_0(Q))$, let $a_n(T)$ denotes $d(T, K_n(X, C_0(Q)))$.*

It is obvious from Lemma 0.1 that there is no problem in introducing the same symbol "a_n" in both cases of Definition 0.2, since $a_n(f)$ is attained for each $f \in C_0(Q, (X^*, \omega^*))$ [resp. $C_0(Q, X^*)$] if and only if $a_n(T)$ is attained for each $T \in L(X, C_0(Q))$ [resp. $K(X, C_0(Q))$].

1. $\mathcal{K}_n(c, c_0)$ IS NOT PROXIMINAL IN $L(c, c_0)$

In this section it will be shown that for each positive integer $n \geqslant 1$, the set $K_n(c, c_0)$ is not proximinal in $L(c, c_0)$. By Lemma 0.1 it is enough to show that for each positive integer $n \geqslant 1$, there is a bounded sequence $\{x_i\}_{i=1}^{\infty}$ in c^*, that converges to zero with respect to the ω^*-topology on c^*, and $a_n(\{x_i\}_{i=1}^{\infty})$ is not attained, that is there is no bounded sequence $\{\tau_i\}_{i=1}^{\infty}$ in any n-dimensional subspace of c^* such that $\tau_i \to 0$ and $\|\{x_i\}_{i=1}^{\infty} - \{\tau_i\}_{i=1}^{\infty}\| = a_n(\{x_i\}_{i=1}^{\infty})$.

The first step in the proof is to find an n-dimensional subspace N_0 of l_1, and finite subset D of l_1, such that N_0 is the unique extremal subspace for $d_n(D, l_1)$, that is, $d_n(D, l_1) = \delta(D, N_0)$, and for any n-dimensional subspace $N \neq N_0$ in l_1, $\delta(D, N) > d_n(D, l_1)$. This will be done in Lemma 1.5. The second step is to find a bounded sequence $\{y_i\}_{i=1}^{\infty}$ in l_1 that satisfies certain conditions, and such that $d\{y_i\}_{i=1}^{\infty}, C_0(N_0))$ is not attained. This will be done in Lemma 1.6. In Lemma 1.7, the set D and the sequence $\{y_i\}_{i=1}^{\infty}$ will be used together to obtain the required result.

1.1. DEFINITION. The c-topology on l_1 is the topology for which a bounded sequence $\{x^k\}_{k=1}^{\infty}$ in l_1 converges to zero, iff for each $y = (y_1, y_2, ...,) \in c$ $\lim_{k \to \infty} [x_1^k \lim y_i + \sum_{i=2}^{\infty} x_i^k y_{i-1}] = 0$, where $x^k = (x_1^k, x_2^k, ...,)$.

1.2. PROPOSITION. *For each* $x=(x_1, x_2,\ldots)\in l_1$, *define the linear functional* $\chi \in C^*$ *by* $\chi(y)=x_1 \lim y_i + \sum_{i=2}^{\infty} x_i y_{i-1}$, $y=(y_1, y_2,\ldots)\in c$. *Under this identification* l_1 *is isometric to* c^*, *and the c-topology on* l_1 *corresponds to the* ω^**-topology on* c^*.

1.3. PROPOSITION (Brown [1]). *Let B be an* $(n+1)$*-dimensional normed linear space and let L be an n-dimensional subspace of B. There is a subset A of B consisting of* $(n+1)$ *points, such that* $d_n(A, B)>0$ *and L is the unique extremal n-dimensional subspace of B for* $d_n(A, B)$.

1.4.

Let $\{e_i'\}_{i=1}^{n+1}$ be the standard basis in l_{n+1}^1, that is $e_i'=(0, 0,\ldots, 0, 1, 0,\ldots, 0)$, and let N_0' be the subspace generated by $\{e_i'\}_{i=1}^{n}$. By Proposition 1.3, there is a subset A of l_{n+1}^1 consisting of a finite number of elements, such that $d_n(A, l_{n+1}^1)=2$, and N_0' is the unique extremal n-dimensional subspace for $d_n(A, l_{n+1}^1)$. Let $x_i=4e_i'+2e_{n+1}'=(0,\ldots, 0, 4, 0,\ldots, 0, 2)\in l_{n+1}^1$, $i=1, 2,\ldots, n$. Then $d(x_i, N_0')=2$, and for each $i\geqslant 1$, the element $y_i=4e_i'$ is the unique element in N_0' such that $\|x_i-y_i\|=d(x_i, N_0')$. Let $A'=A\cup\{x_1,\ldots, x_n\}$, then

(1) A' consists of a finite number of elements,

(2) $d_n(A', l_{n+1}^1)=\delta(A', N_0')=2$,

(3) N_0' is the unique extremal subspace for $d_n(A', l_{n+1}^1)$, and

(4) for each $i=1, 2,\ldots, n$ the element $y_i=4e_i'$ is the unique element in N_0' such that $\|x_i-y_i\|\leqslant 2$.

1.5. LEMMA. *Let* $\{e_i\}_{i=1}^{\infty}$ *be the standard basis in* l_1, *that is* $e_i=(0, 0,\ldots, 0, 1, 0,\ldots)$. *Let* $\gamma_0=\sum_{i=n+2}^{\infty} a_i e_i$ *be an element in* l_1, *let* i_0 *be a positive integer such that,* $1\leqslant i_0\leqslant n$ *and let* N_0 *be the n-dimensional subspace of* l_1 *generated by*

$$\{e_1,\ldots, e_{i_0-1}, e_{i_0}+\gamma_0, e_{i_0+1},\ldots, e_n\}.$$

There is a subset D of l_1, *consisting of a finite number of elements, such that*

(1) $d_n(D, l_1)=\delta(D, N_0)=2$,

(2) N_0 *is the unique extremal subspace of* l_1 *for* $d_n(D, l_1)$.

Proof. Let A', N_0', $\{x_i\}_{i=1}^{n}$ and $\{e_i'\}_{i=1}^{n+1}$ be as in 1.4. For $x'=\sum_{i=1}^{n+1}\alpha_i e_i'$ in A' one can choose $y'=\sum_{i=1}^{n}\lambda_i e_i'\in N_0'$, such that $\|x'-y'\|\leqslant 2$. Define $\psi(x')=\sum_{i=1}^{n+1}\alpha_i e_i+\lambda_{i_0}\gamma_0\in l_1$. Then

$$\left\|\psi(x')-\left(\sum_{i=1}^{n}\lambda_i e_i+\lambda_{i_0}(e_{i_0}+\gamma_0)\right)\right\|\leqslant 2.$$

Let $D=\{\psi(x'); x'\in A\}$, then D consists of a finite number of elements and

$\delta(D, N_0)=2$. To complete the proof it will be shown that if N is an n-dimensional subspace of l_1 and $\delta(D, N)\leqslant 2$ then $N=N_0$. Let $P: l_1 \to l^1_{n+1}$ be defined by

$$P((x_i)_{i=1}^{\infty})=(x_i)_{i=1}^{n+1}.$$

Clearly $P(D)=A'$, and since $\delta(D, N)\leqslant 2$ then $\delta(A', P(N))\leqslant 2$. Thus by 1.4 $P(N)=N'_0$, therefore N has a basis of the form

$$d_i=e_i+\sum_{k=n+2}^{\infty}\beta^i_k e_k, \qquad i=1, 2,..., n.$$

Also by 1.4 for each $i=1, 2,..., n$, $x_i=4e'_i+2e'_{n+1}\in l^1_{n+1}$ is contained in A' and $4e'_i$ is the unique element in N'_0 that approximates x_i, so

$$\psi(x_i)=4e_i+2e_{n+1} \qquad \text{for } i\neq i_0$$

and

$$\psi(x_{i_0})=4e_{i_0}+4\gamma_0+2e_{n+1}.$$

Using the fact that $p(N)=N'_0$, if $z_i\in N$ approximates $\psi(x_i)$, then

$$z_i=4e_i \qquad \text{for } i\neq i_0$$

and

$$z_{i_0}=4e_{i_0}+4\gamma_0.$$

Therefore $N_0\subseteq N$, and since $\dim N_0=\dim N=n$, it follows that $N_0=N$. ∎

1.6. Lemma. *Let $\{e_i\}_{i=1}^{\infty}$ be the standard basis in l_1. Let*

$$\gamma_0=\sum_{i=n+2}^{\infty}\frac{e_i}{2^{i-n-1}}=(\underbrace{0,0,...,0}_{(n+1)\text{ times}}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8},...)\in l_1$$

and let N_0 be the n-dimensional subspace of l_1 generated by

$$\{-e_1+\gamma_0, e_2, e_3,..., e_n\}.$$

There is a bounded sequence $\{y_i\}_{i=1}^{\infty}$ in l_1 with the following properties:

(1) *$\{y_i\}_{i=1}^{\infty}$ converges to zero with respect to the c-topology on l_1,*

(2) *$d(\{y_i\}_{i=1}^{\infty}, c_0(N_0))=2$, and*

(3) *$d(\{y_i\}_{i=1}^{\infty}, c_0(N_0))$ is not attained.*

Proof. Let $\alpha_0=-e_1+\sum_{i=2}^{\infty}e_i/2^{i-1}=(-1, \frac{1}{2}, \frac{1}{4},...)\in l_1$, and let M be the one-dimensional subspace generated by α_0.

For each $k \geqslant 1$ let

$$\psi_k = -e_1 + \sum_{i=k}^{\infty} \frac{e_i}{2^{i-k+1}} = (-1, \underbrace{0, 0, ..., 0}_{(k-2)\text{times}}, \tfrac{1}{2}, \tfrac{1}{4}, ...).$$

It will be shown that for $k \geqslant 3$ the point α_0 is the unique best approximation to ψ_k from M. Assume that $d(\psi_k, M) = \|\lambda\alpha_0 - \psi\|$, then

$$\begin{aligned}\|\lambda\alpha_0 - \psi_k\| &= |\lambda - 1| + \sum_{i=2}^{k-1} \frac{|\lambda|}{2^{i-1}} + \sum_{i=k}^{\infty} \left|\frac{\lambda}{2^{i-1}} - \frac{1}{2^{i-k+1}}\right| \\ &= |\lambda - 1| + |\lambda|\left(1 - \frac{1}{2^{k-2}}\right) + \left|\frac{2^{k-2} - \lambda}{2^{k-2}}\right|.\end{aligned}$$

The only local minimum points for this equation occur when $\lambda = 1$, $\lambda = 0$ and $\lambda = 2^{k-2}$. If $\lambda = 1$ then $\|\lambda\alpha_0 - \psi_k\| = 1 - 2^{-k+2} + 1 - 2^{-k+2} = 2 - 2^{-k+3}$, if $\lambda = 0$ then $\|\lambda\alpha_0 - \psi_k\| = \|\psi_k\| = 2$, and if $\lambda = 2^{k-2}$ then $\|\lambda\alpha_0 - \psi_k\| = 2^{k-2} - 1 + 2^{k-2} - 1 = 2^{k-1} - 2$. So for $k \geqslant 3$, $\min_{\lambda \in R} \|\lambda\alpha_0 - \psi_k\| = \|\alpha_0 - \psi_k\|$, that is α_0 is the unique best approximation to ψ_k from M. For each positive integer $k \geqslant 3$, let $a_k = 2/\|\alpha_0 - \psi_k\|$. Since $\|\alpha_0 - \psi\| < 2$ and $\|\alpha_0 - \psi\| \to 2$, it follows that $a_k > 1$ and $a_k \to 1$. Let $\phi_k = a_k\psi_k$. Then since $\|\psi_k\| = 2$, it follows that $\|\phi_k\| > 2$, $d(\phi_k, M) = 2$ and $a_k\alpha_0$ is the unique best approximation to ϕ_k from M when $k \geqslant 3$. Furthermore

(a) Since $\{\psi_k\}_{k=3}^{\infty}$ converges to zero with respect to the c-topology on l_1, it follows that $\{\phi_k\}_{k=3}^{\infty}$ converges to zero with respect to the same topology.

(b) It will be shown that $d(\{\phi_k\}_{k=3}^{\infty}, c_0(M))$ is not attained.

Let $\varepsilon > 0$ be given, and let k_0 be so that for all $k \geqslant k_0$

$$\|\phi_k\| \leqslant 2 + \varepsilon.$$

Define $\{\tau_k\}_{k=3}^{\infty}$ in M as follows:

$$\tau_k = \begin{cases} a_k\alpha_0 & \text{if } k \leqslant k_0 \\ 0 & \text{if } k > k_0 \end{cases}$$

Then $\{\tau_k\}_{k=3}^{\infty} \in c_0(M)$ and $\|\{\phi_k\}_{k=3}^{\infty} - \{\tau_k\}_{k=3}^{\infty}\| \leqslant 2 + \varepsilon$. Thus $d(\{\phi_k\}_{k=3}^{\infty}, c_0(M)) \leqslant 2$. But the only sequence $\{\tau_k\}_{k=3}^{\infty}$ in M for which $\|\{\phi_k\}_{k=3}^{\infty} - \{\tau_k\}_{k=3}^{\infty}\| = 2$ is $\{\tau_k\}_{k=3}^{\infty} = \{a_k\alpha_0\}_{k=3}^{\infty}$, and since $a_k \to 1$, it follows that $\tau_k \nrightarrow 0$. So $d(\{\phi_k\}_{k=3}^{\infty}, c_0(M))$ is not attained. Define $P\colon l_1 \to l_1$ by

$$P(x_i) = x_1 e_1 + \sum_{i=2}^{\infty} x_i e_{n+i} = (x_1, \underbrace{0, 0, ..., 0}_{n-\text{times}}, x_2, x_3, ...).$$

Then P is an isometry from l_1 into l_1 and $P(\alpha_0) = -e_1 + \gamma_0$. Let y_1, y_2 be any two elements in N_0, and for $k \geqslant 3$, let $y_k = P(\phi_k)$. Then the sequence $\{y_k\}_{k=1}^{\infty}$ converges to zero with respect to the c-topology on l_1. Furthermore if $k \geqslant 3$ and $x = c_1(-e_1 + \gamma_0) + \sum_{i=2}^{n} c_i e_i$ in N_0, then

$$\begin{aligned} \| y_k - x \| &= \sum_{i=2}^{n} |c_i| + \| y_k - c_1(-e_1 + \gamma_0) \| \\ &\geqslant \| y_k - c_1(-e_1 + \gamma_0) \| \\ &= \| P(\phi_k) - c_1 P(\alpha_0) \| \\ &= \| \phi_k - c_1 \alpha_0 \| . \end{aligned}$$

Therefore for $k \geqslant 3$, the element $a_k(-e_1 + \gamma_0)$ is the unique best approximation to y_k from N_0. Thus as in (b) one can show that $d(\{y_k\}_{k=1}^{\infty}, c_0(N_0)) = 2$, and it is not attained. ▮

1.7. Lemma. *For each positive integer $n \geqslant 1$, there is a bounded sequence $\{x_k\}_{k=1}^{\infty}$ in l_1, such that $\{x_k\}_{k=1}^{\infty}$ converges to zero with respect to the c-topology on l_1 and $a_n(\{x_k\}_{k=1}^{\infty}$ is not attained.*

Proof. Let N_0 and γ_0 be as in Lemma 1.6. By Lemma 1.5 "taking $i_0 = 1$ and replace γ_0 by $-\gamma_0$" there is a subset D of l_1 consisting of finite number of elements, such that $d_n(D, l_1) = \delta(D, N_0) = 2$, and N_0 is the unique extremal n-dimensional subspace for $d_n(D, l_1)$. Without loss of generality let $D = \{z_1, ..., z_m\}$, and let $\{y_k\}_{k=1}^{\infty}$ be as in Lemma 1.6. Define the sequence $\{x_k\}_{k=1}^{\infty}$ in l_1 as follows:

$$x_k = \begin{cases} z_k & \text{for } k = 1, 1, ..., m \\ y_{k-m} & \text{for } k = m+1, m+2, ... \end{cases}$$

Then $\{x_k\}_{k=1}^{\infty}$ converges to zero with respect to the c-topology on l_1, and $a_n(\{x_k\}_{k=1}^{\infty}) \leqslant d(\{x_k\}_{k=1}^{\infty}, c_0(N_0)) = 2$. Assume that there is an n-dimensional subspace N of l_1, and a sequence $\{\tau_k\}_{k=1}^{\infty}$ in N such that $\| \{x_k\}_{k=1}^{\infty} - \{\tau_k\}_{k=1}^{\infty} \| \leqslant 2$, then $\delta(D, N) \leqslant 2$ so by lemma 1.5, $N = N_0$, and thus by Lemma 1.6 $\tau_k \nrightarrow 0$. So $a_n(\{x_k\}_{k=1}^{\infty}$ is not attained. ▮

1.8. Theorem. *For each positive integer $n \geqslant 1$ the set $K_n(c, c_0)$ is not proximinal in $L(c, c_0)$.*

Proof. Follows from Lemma 0.1, Proposition 1.2, and Lemma 1.7. ▮

Theorem 1.8 gives a negative solution to Problem 5.2.2 in Deutsch, Mach and Saatkamp [2] when $X = c$.

2. $K_n(E, c)$ IS NOT PROXIMINAL IN $K(E, c)$ FOR $E = c_0$ AND c

In this section it will be shown that for $E = c_0$ and c and for each positive integer $n \geqslant 1$, the set $K_n(E, c)$ is not proximinal in $K(E, c)$. The argument of the proof is similar to that one in Section 1 and the main step is Lemma 2.1.

2.1. LEMMA. *Let $\{e_i\}_{i=1}^{\infty}$ be the standard basis in l_1, let*

$$\alpha_0 = e_1 + e_3 + \sum_{k=1}^{\infty} \frac{1}{2^k} e_{2k+2} + \sum_{k=1}^{\infty} \frac{1}{2^k} e_{2k+3}$$
$$= (1, 0, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \ldots) \in l_1$$

and let M be the one-dimensional subspace of l_1 generated by α_0. There is a bounded sequence $\{\beta_i\}_{i=1}^{\infty}$ in l_1 satisfying the following properties:

(1) *$\{\beta_i\}_{i=1}^{\infty}$ converges with respect to the norm-topology on l_1,*

(2) *$d(\{\beta_i\}_{i=1}^{\infty}, c(M)) = 2$, and*

(3) *$d(\{\beta_i\}_{i=1}^{\infty}, c(M))$ is not attained.*

Proof.
Let

$$\beta_0 = (1/2e_1) - (1/2e_3) + \sum_{k=1}^{\infty} (1/2^{k+1})\, e_{2k+2} - \sum_{k=1}^{\infty} (1/2^{k+1})\, e_{2k+3}$$
$$= (\tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{8}, -\tfrac{1}{8}, \ldots) \in l_1$$

and let $\{\psi_i\}_{i=1}^{\infty}$ be the sequence in l_1, defined as follows:

$$\psi_{2i-1} = \beta_0 + \frac{1}{2^i} e_{2i+3}, \quad i = 1, 2, 3, \ldots,$$

$$\psi_{2i} = \beta_0 - \frac{1}{2^i} e_{2i+2}, \quad i = 1, 2, 3, \ldots.$$

Clearly $\psi_i \to \beta_0$. It will be shown that for each $i \geqslant 1$ the element $\frac{1}{2}\alpha_0$ is the unique best approximation for ψ_{2i-1} from M, and $-\frac{1}{2}\alpha_0$ is the unique best approximation for ψ_{2i} from M. Let $i \geqslant 1$ be a fixed positive integer. For any real number λ

$$\|\lambda\alpha_0 - \psi_{2i-1}\|$$
$$= \left\| \left[\lambda(e_1 + e_3) + \sum_{k=1}^{\infty} \frac{\lambda}{2^k} (e_{2k+2} + e_{2k+3}) \right] \right.$$

$$-\left[\frac{1}{2}(e_1-e_3)+\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}(e_{2k+2}-e_{2k+3})+\frac{1}{2^i}e_{2i+3}\right]\Bigg\|$$

$$=\left|\lambda-\frac{1}{2}\right|+\left|\lambda+\frac{1}{2}\right|+\left|\lambda-\frac{1}{2}\right|\cdot\left(\sum_{k=1}^{\infty}\frac{1}{2^k}\right)+\left|\lambda+\frac{1}{2}\right|$$

$$\cdot\left(\sum_{\substack{k=1\\k\neq i}}^{\infty}\frac{1}{2^k}\right)+\left|\frac{\lambda-(1/2)}{2^i}\right|$$

$$=\left|\lambda-\frac{1}{2}\right|\left(2+\frac{1}{2^i}\right)+\left|\lambda+\frac{1}{2}\right|\left(2-\frac{1}{2^i}\right).$$

It follows from this that $\|\lambda\alpha_0-\psi_{2i-1}\|$ is minimum only when $\lambda=\frac{1}{2}$. In the same way one can show that

$$\|\lambda\alpha_0-\psi_{2i}\|=|\lambda-\frac{1}{2}|\left(2-\frac{1}{2^i}\right)+|\lambda+\frac{1}{2}|\left(2+\frac{1}{2^i}\right),$$

which is minimum only when $\lambda=-\frac{1}{2}$. Thus for each positive integer $i\geqslant 1$,

$$d(\psi_{2i-1},M)=\left\|\psi_{2i-1}-\frac{1}{2}\alpha_0\right\|=2-\frac{1}{2^i}<2,$$

$$d(\psi_{2i},M)=\left\|\psi_{2i}+\frac{1}{2}\alpha_0\right\|=2-\frac{1}{2^i}<2,$$

and

$$\|\psi_{2i-1}\|=\|\psi_{2i}\|=2.$$

For each positive integer $k\geqslant 1$, let $\lambda_k=2/d(\psi_k,M)$, then $\lambda_k>1$ and $\lambda_k\to 1$. Let $\beta_k=\lambda_k\psi_k$, then

(1) Since $\psi_k\to\beta_0$ and $\lambda_k\to 1$, it follows that $\beta_k\to\beta_0$.

(2) It is obvious that $\|\beta_k\|\to 2$, and for each positive integer $k\geqslant 1$, $\|\beta_k\|>2$, so let $\varepsilon>0$ be given, and let $i_0\geqslant 1$ be such that for $i\geqslant i_0$, $\|\beta_i\|\leqslant 2+\varepsilon$.

Define the sequence $\{\tau_k\}_{k=1}^{\infty}$ in M as follows:

$$\tau_k=\begin{cases}\text{the unique best approximimation for } \beta_k \text{ from } M, & \text{if } k\leqslant i_0\\ 0 & \text{if } k>i_0.\end{cases}$$

Then $\{\tau_k\}_{k=1}^{\infty}\in c(M)$, and $\|\{\beta_k\}_{k=1}^{\infty}-\{\tau_k\}_{k=1}^{\infty}\|\leqslant 2+\varepsilon$. Thus

$$d(\{\beta_k\}_{k=1}^{\infty},c(M))\leqslant 2.$$

(3) The only sequence $\{\tau_k\}_{k=1}^{\infty}$ in M satisfies the inequality

$$\|\{\beta_k\}_{k=1}^{\infty}-\{\tau_k\}_{k=1}^{\infty}\|\leqslant 2,$$

is the following sequence:

$$\tau_{2i-1}=\tfrac{1}{2}\lambda_{2i-1}\alpha_0, \qquad i=1,2,...,$$
$$\tau_{2i}=-\tfrac{1}{2}\lambda_{2i}\alpha_0, \qquad i=1,2,...,$$

which is not in $c(M)$. ∎

2.2. LEMMA. *For each positive integer $n\geqslant 1$, there is a convergent sequence $\{x_i\}_{i=1}^{\infty}$ in l_1 such that $a_n(\{x_i\}_{i=1}^{\infty})$ is not attained.*

Proof. Let $\{e_i\}_{i=1}^{\infty}$, α_0 and $\{\beta_i\}_{i=1}^{\infty}$ be as in Lemma 2.1, let

$$\gamma_0=e_{n+2}+\sum_{k=1}^{\infty}\frac{1}{2^k}e_{n+2k+1}+\sum_{k=1}^{\infty}\frac{1}{2^k}e_{n+2k+2}$$
$$=(\underbrace{0,...,0}_{(n+1)\text{times}},1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4},...)\in l_1,$$

and let N_0 be the n-dimensional subspace of l_1 generated by $\{e_1,e_2,...,e_{n-1},e_n+\gamma_0\}$. Define $T\colon l_1\to l_1$ by

$$T((x_i)_{i=1}^{\infty})=\sum_{i=1}^{\infty}x_ie_{n+i-1}=(\underbrace{0,...,0}_{(n-1)\text{times}},x_1,x_2,...).$$

T is an isometry from l_1 into l_1, and $T(\alpha_0)=e_n+\gamma_0$. Let $y_i=T(\beta_i)$ $i=1,2,...$. Then $\{y_i\}_{i=1}^{\infty}$ is a convergent sequence in l_1, and $d(\{y_i\}_{i=1}^{\infty}, c(N_0))=2$. The only sequence $\{\psi_i\}_{i=1}^{\infty}$ in N_0 satisfying $\|\{y_i\}_{i=1}^{\infty}-\{\psi_i\}_{i=1}^{\infty}\|=2$ is the image under the isometry T of the unique sequence $\{\tau_i\}_{i=1}^{\infty}$ in M satisfying $\|\{\beta_i\}_{i=1}^{\infty}-\{\tau_i\}_{i=1}^{\infty}\|=2$. But by Lemma 2.1 this sequence does not converge. So $d(\{y_i\}_{i=1}^{\infty}, c(N_0))$ is not attained. By Lemma 1.5 "taking $i_0=n$," there is a subset D of l_1 consisting of finite number of elements, such that $d_n(D,l_1)=\delta(D,N_0)=2$, and N_0 is the unique extremal n-dimensional subspace for $d_n(D,l_1)$. Let $D=\{z_1,...,z_m\}$ and define the sequence $\{x_i\}_{i=1}^{\infty}$ in l_1 as follows:

$$x_i=\begin{cases} z_i & \text{for } i=1,2,...,m\\ y_{i-m} & \text{for } i=m+1,m+2,....\end{cases}$$

One can easily show that $\{x_i\}_{i=1}^{\infty}$ is a convergent sequence in l_1 and $a_n(\{x_i\}_{i=1}^{\infty})$ is not attained. ∎

2.3. COROLLARY. *If $E=c$ or c_0 then for each positive integer $n\geqslant 1$ the set $K_n(E,c)$ is not proximinal in $K(E,c)$.*

Proof. Follows from Lemmas 0.1 and 2.2.

3. The Proximinality of $K_n(C_0(Q), C_0(S))$ in $K(C_0(Q), C_0(S))$

In this section the proximinality of $K_n(C_0(Q), C_0(S))$ in $K(C_0(Q), C_0(S))$ and in $L(C_0(Q), C_0(S))$ will be studied in detail. It will be shown that if Q and S are locally compact Hausdorff spaces, and S contains Q_0, then $K_n(C_0(Q), C_0(S))$ is proximinal in $K(C_0(Q), C_0(S))$ iff Q is finite. It will be shown also that if Q is a locally compact Hausdorff space, that contains Q_0 then $K_n(C_0(Q), c_0)$ is not proximinal in $L(C_0(Q), c_0)$. The first step in the proof is to show that if X is a Banach space, and Q is a locally compact Hausdorff space that contains Q_0, then the proximinality of $K_n(X, c)$ in $K(X, c)$ [resp. $L(X, c)$] is a necessary condition for the proximinality of $K_n(X, C_0(Q))$ in $K(X, C_0(Q))$ [resp. $L(X, C_0(Q))$]. This will be established in Lemma 3.4. The second step is to show that if Q is a locally compact Hausdorff space, Y is a closed subset of Q and X is a Banach space then the proximinality of $K_n(C_0(Y), X)$ in $K(C_0(Y), X)$ [resp. $L(C_0(Y), X)$], is a necessary condition for the proximinality of $K_n(C_0(Q), X)$ in $K(C_0(Q), X)$ [resp. $L(C_0(Q), X)$]. This will be established in Lemma 3.5. Finally the results of Sections 1 and 2 will be used in Theorems 3.6 and 3.8., to obtain the main results.

The closed subspace Y of the Banach space X is called a norm-one-complemented subspace of X if there is a linear projection $P: X \to Y$ such that $\|P\| = 1$. The proof of the following proposition depends on this property:

3.1. Proposition. *Let Q be a locally compact Hausdorff space, E a Banach space and F a norm-one-complemented subspace of E. If $a_n(f)$ is attained in E for each $f \in C_0(Q, E)$, then $a_n(g)$ is attained in F for each $g \in C_0(Q, F)$.*

The proof of the following lemma is straight forward:

3.2. Lemma. *Let Q be a locally compact Hausdorff space and let $X = C_0(Q)$. If Q is infinite then there is a subspace Y of X^* satisfying the following properties:*

(1) *Y is isometrically isomorphic to l_1.*

(2) *Y is a norm-one-complemented subspace of X^*.*

Lemma 3.3 is similar to the Extension of Tietze's Theorem due to Dugundji [3].

3.3. Lemma. *Let X be a Banach space, Q a locally compact Hausdorff space that contains Q_0 and let $\{b_i\}_{i=1}^{\infty}$ be an infinite sequence of distinct elements in Q, that converges to b_0 in Q. There is a sequence $\{\phi_i\}_{i=0}^{\infty}$ of real-valued functions on Q with the following properties:*

(a) *For* $i=1, 2, 3,\dots$, *the function* ϕ_i *is continuous.*

(b) $0 \leqslant \phi_i(q) \leqslant 1$, *for all* $q \in Q$ *and* $i=0, 1, 2,\dots$

(c) $\phi_i(b_j)=\delta_{ij}$ *for* $i=1, 2, 3,\dots$, *and* $j=1, 2, 3,\dots$

(d) $\phi_0(b_i)=0$ *for* $i=1, 2, 3,\dots$

(e) $\sum_{i=0}^{\infty} \phi_i(q) \leqslant 1$ *for all* $q \in Q$.

(f) *There is a compact subset* Y *of* Q, *such that* $\sum_{i=0}^{\infty} \phi_i \equiv 0$ *outside* Y.

(g) *If* $\{x_i\}_{i=1}^{\infty}$ *is a bounded sequence in* X *that converges to* x_0, *then the function* $f\colon Q \to X$ *defined by* $f(q)=\sum_{i=0}^{\infty} \phi_i(q)\, x_i$, *is an element in* $C_0(Q, X)$.

(h) *If* X *is a dual space and* $\{x_i\}_{i=1}^{\infty}$ *is a bounded sequence in* X, *that is,* ω^**-convergent to* x_0 *then the function* $f\colon Q \to X$ *defined by* $f(q)=\sum_{i=0}^{\infty} \phi_i(q)\, x_i$, *is an element in* $C_0(Q, (X, \omega^*))$.

Proof. Since $b_i \to b_0$ one can show that there is a relatively compact open subset U of Q, such that $\{b_i\}_{i=0}^{\infty} \subseteq U$. Let $Y=\bar{U}$, and let $g\colon Q \to R$ be a continuous function with the following properties:

(1) $g(q)=0$ for $q \notin U$, and $\|g\|=1$,

(2) $g(b_i)=1$ for $i=1, 2,\dots$, and $g(q) \geqslant 0$ for all $q \in Q$.

Let $\{V_1, U_1\}$ be an open cover for Q that satisfies the properties that $V_1 \cap \{b_i\}_{i=0}^{\infty}=\{b_1\}$, and $b_1 \notin U_1$. Let $\{\phi'_1, g'_1\}$ be a partition of unity corresponding to $\{V_1, U_1\}$. Then $\phi'_1(b_1)=1$, $\phi_1(b_i)=0$ for $i \neq 1$, $g'_1(b_1)=0$, and $g'_1(b_i)=1$ for $i \neq 1$. Let $\phi_1=\phi'_1 \cdot g$ and let $g_1=g'_1 \cdot g$, then $\phi_1+g_1=g$. By the same method, for each $n \geqslant 1$, "by taking g_{n-1} in place of g," one can show that there are two nonnegative continuous functions ϕ_n and g_n with the properties that $\phi_n(b_n)=1$, $\phi_n(b_i)=0$ for $i \neq n$ and $\phi_n+g_n=g_{n-1}$, that is, $\sum_{i=1}^{n} \phi_i+g_n=g$. Since $\{\phi_i\}_{i=1}^{n}$ and g_n are nonnegative, it follows that for each $q \in Q$, $\sum_{i=1}^{n} \phi_i(q) \leqslant g(q)$. Thus by induction there are two bounded sequences $\{\phi_i\}_{i=1}^{\infty}$ and $\{g\}_{i=1}^{\infty}$ of nonnegative continuous functions on Q, satisfying that $\phi_i(b_j)=\delta_{ij}$, $0 \leqslant \phi_i(q) \leqslant 1$ for each $q \in Q$, $\phi_i \equiv 0$ outside Y, and $\sum_{i=1}^{n} \phi_i+g_n=g$. Clearly $\sum_{i \leqslant 1}^{\infty} \phi_i(q) \leqslant g(q)$ for each $q \in Q$, so let $\phi_0=g-\sum_{i=1}^{\infty} \phi_i$. Then $0 \leqslant \phi_0(q) \leqslant 1$ for each $q \in Q$, $\phi_0 \equiv 0$ outside Y, and $\phi_0(b_i)=0$ for $i=1, 2,\dots$ Thus the conditions (a)–(f) are satisfied.

(g) Assume that $x_i \to x_0$ in X, and let $f\colon Q \to X$ be defined by

$$f(q)=\phi_0(q)\, x_0+\sum_{i=1}^{\infty} \phi_i(q)\, x_i \qquad \text{for } q \in Q.$$

Then

$$f(q)=\left[\phi_0(q)+\sum_{i=1}^{\infty}\phi_i(q)\right]\cdot x_0+\sum_{i=1}^{\infty}\phi_i(q)\cdot[x_i-x_0]$$

$$=g(q)\,x_0+\sum_{i=1}^{\infty}\phi_i(q)\cdot[x_i-x_0].$$

Since for each $i\geqslant 1$ the function ϕ_i is continuous, and since $\|x_i-x_0\|\to 0$, it follows that the function $\sum_{i=1}^{\infty}\phi_i(q)\cdot[x_i-x_0]$ is continuous. Thus the function f is continuous, and since $f\equiv 0$ outside y then $f\in C_0(Q, X)$.

The proof of (h) is similar to that of (g). ▮

3.4. LEMMA. *Let X be a Banach space, and let Q be a locally compact Hausdorff space that contains Q_0. If $K_n(X, c)$ is not proximinal in $K(X, c)$ [resp. $L(X, c)$], then $K_n(X, C_0(Q))$ is not proximinal in $K(X, C_0(Q))$ [resp. $L(X, C_0(Q)$].*

Proof. Assume that $K_n(X, c)$ is not proximinal in $K(X, c)$ [resp. $L(X, c)$]. Then by Lemma 0.1 there is a convergent [resp. ω^*-convergent] sequence $\{x_i\}_{i=1}^{\infty}$ in X^* such that

$$a_n(\{x_i\}_{i=1}^{\infty})=\inf\{d(\{x_i\}_{i=1}^{\infty}, c(N)); \dim N\leqslant n, N\subseteq X^*\}$$

is not attained.

Since Q contains Q_0, it follows that there is an infinite sequence $\{b_i\}_{i=1}^{\infty}$ of distinct elements in Q, that converges to some point b_0 in Q. As in Lemma 3.3, let $\{\phi_i\}_{i=0}^{\infty}$ be a sequence of nonnegative fuctions defined on Q, corresponding to $\{b_i\}_{i=1}^{\infty}$. Define $f\colon Q\to X^*$ by $f(q)=\sum_{i=0}^{\infty}\phi_i(q)\,x_i$. Then by Lemma 3.3, $f\in C_0(Q, X^*)$ [resp. $f\in C_0(Q, X^*, \omega^*))$], and $f(b_i)=x_i$ for each $i=1, 2,\ldots$. It will be shown that $a_n(f)=a_n(\{x_i\}_{i=1}^{\infty})$. Let $g\colon Q\to X^*$ be a continuous function with $g(Q)\subseteq N$ for some n-dimensional N of X^*, and let $y_i=g(b_i)$. Since $b_i\to b_0$, then the sequence $\{y_i\}_{i=1}^{\infty}$ converges to $y_0=g(b_0)$, and

$$\|\{x_i\}_{i=1}^{\infty}-\{y_i\}_{i=1}^{\infty}\|=\sup_i\|f(b_i)-g(b_i)\|\leqslant\|f-g\|.$$

Therefore $a_n(\{x_i\}_{i=1}^{\infty})\leqslant a_n(f)$. Second, let $\{y_i\}_{i=1}^{\infty}$ be a convergent sequence in an n-dimensional subspace N of X^*, and define $g\colon Q\to N$ by $g(q)=\sum_{i=0}^{\infty}\phi_i(q)\,y_i$. By Lemma 3.3, $g\in C_0(Q, N)$, and for each $q\in Q$

$$\|(f-g)(q)\|=\left\|\sum_{i=0}^{\infty}\phi_i(q)(x_i-y_i)\right\|\leqslant\sum_{i=0}^{\infty}\phi_i(q)\|x_i-y_i\|$$

$$\leqslant\sup_i\|x_i-y_i\|=\|\{x_i\}_{i=1}^{\infty}-\{y_i\}_{i=1}^{\infty}\|.$$

Thus $\|f-g\|\leqslant\|\{x_i\}_{i=1}^{\infty}-\{y_i\}_{i=1}^{\infty}\|$, therefore $a_n(f)\leqslant a_n(\{x_i\}_{i=1}^{\infty})$. It is clear from the first part of the proof, that if $a_n(f)$ is attained then

$a_n(\{x_i\}_{i=1}^{\infty})$ is attained. But $a_n(\{x_i\}_{i=1}^{\infty})$ is not attained, so $a_n(f)$ is not attained. ∎

3.5. LEMMA. *Let Q be a locally compact Hausdorff space, Y a closed subset of Q and let X be a Banach space. If $K_n(C_0(Y), X)$ is not proximinal in $K(C_0(Y), X)$* [resp. $L(C_0(Y), X)$] *then $K_n(C_0(Q), X)$ is not proximinal in $K(C_0(Q), X)$ [resp. $L(C_0(Q), X)$].*

Proof. Let $P: C_0(Q) \to C_0(Y)$ be defined by

$$P(f) = f_{|Y}, f \in C_0(Q).$$

P is a linear and onto mapping with $\| P(f)\| \leqslant \| f \|$ for each $f \in C_0(Q)$. Let T be an operator in $K(C_0(Y), X)$ [resp. $L(C_0(Y), X)$], then $T' = T \circ P$ is an operator in $K(C_0(Q), X)$ [resp. $L(C_0(Q), X)$]. It will be shown that $a_n(T) = a_n(T')$, and if $a_n(T)$ is not attained then $a_n(T')$ is not attained. If $K: C_0(Y) \to X$ is a bounded linear operator of rank less than or equal to n, then $K' = K \circ P$ is an operator in $K_n(C_0(Q), X)$. Furthermore

$$\| T' - K' \| \leqslant \| P \| \, \| T - K \| = \| T - K \|.$$

Thus $a_n(T') \leqslant a_n(T)$.
Second, let $K' \in K_n(C_0(Q), X)$, then there are $\{\mu_k\}_{k=1}^{n}$ in $(C_0(Q))^*$, and $\{x_k\}_{k=1}^{n}$ in X, such that for each $f \in C_0(Q)$

$$K'(f) = \sum_{k=1}^{n} \mu_k(f) \cdot x_k.$$

For each $k = 1, 2, \ldots, n$, let $\mu'_k = \mu_{k|Y}$, and let $K: C_0(Y) \to X$ be defined by

$$K(f) = \sum_{k=1}^{n} \mu'_k(f) \cdot x_k, \qquad f \in C_0(Y).$$

Clearly $K \in K_n(C_0(Y), X)$. It will be shown that for each $f \in C_0(Y)$ with $\| f \| \leqslant 1$, there is a net $\{f_\alpha\}_{\alpha \in I}$ in the unit ball of $C_0(Q)$ such that

$$\| (T - K)(f) \| \in \overline{\{\| (T' - K')(f_\alpha) \|\}_{\alpha \in I}},$$

If this is true then $\| T - K \| \leqslant \| T' - K' \|$, and therefore $a_n(T) = a_n(T')$. Let $\{U_\alpha\}_{\alpha \in I}$ be the family of all open sets in Q that contain Y. For each $\alpha \in I$, there is a continuous function $g_\alpha: Q \to R$ with the following properties:

$$g_\alpha(q) = \begin{cases} 1 & \text{for } q \in Y, \\ 0 & \text{for } q \notin U_\alpha, \end{cases}$$

and

$$0 \leqslant g_\alpha(q) \leqslant 1 \qquad \text{for all } q \in Q.$$

On the other hand by Tietze's Extension Theorem, there is a function $F \in C_0(Q)$, such that $\|F\| \leqslant \|f\|$ and $F_{|Y}=f$. Let $f_\alpha = F \cdot g_\alpha$. Then the net $\{f_\alpha\}_{\alpha \in I}$ is contained in the unit ball of $C_0(Q)$. Furthermore for each $\alpha \in I$, $T'(f_\alpha) = T(f)$, and since Q is a Hausdorff space, it follows that $\bigcap_{\alpha \in I} U_\alpha = Y$, thus for each $k = 1, 2, \ldots, n$, $\mu'_k(f) \in \overline{\{\mu_k(f_\alpha)\}_{\alpha \in I}}$. Hence

$$\|(T-K)(f)\| \in \overline{\{\|(T'-K')(f_\alpha)\|\}_{\alpha \in I}}.$$

It is clear from the proof that if $a_n(T')$ is attained then $a_n(T)$ is attained. ∎

3.6. THEOREM. *If Q is a locally compact Hausdorff space, that contains Q_0 then for each positive integer $n \geqslant 1$, the set $K_n(C_0(Q), c_0)$ is not proximinal in $L(C_0(Q), c_0)$.*

Proof. By Theorem 1.8, the set $K_n(c, c_0)$ is not proximinal in $L(c, c_0)$. Since Q contains Q_0, it contains an infinite convergent sequence $\{b_i\}_{i=1}^{\infty}$ of distinct elements, but then $c = C(\overline{\{b_i\}_{i=1}^{\infty}})$. So by Lemma 3.5, the set $K_n(C_0(Q), c_0)$ is not proximinal in $L(C_0(Q), c_0)$. ∎

Note. Theorem 3.6 is not generally true if Q fails to contain Q_0, indeed by Deutsch *et al.* [2], the set $K_n(c_0, c_0)$ is proximinal in $L(c_0, c_0)$.

3.7. LEMMA. *Let Q be a locally compact Hausdorff space. If Q is infinite then for each positive integer $n \geqslant 1$, the set $K_n(C_0(Q), c)$ is not proximinal in $K(C_0(Q), c)$.*

Proof. By Corollary 2.3, the set $K_n(c, c)$ is not proximinal in $K(c, c)$. Thus by Lemma 0.1, there is a convergent sequence $\{x_i\}_{i=1}^{\infty}$ in l_1, such that $a_n(\{x_i\}_{i=1}^{\infty})$ is not attained. By Lemma 3.2, l_1 is isometric to a norm-one-complemented subspace of $(C_0(Q))^*$, thus by Proposition 3.1, there is a convergent sequence $\{y_i\}_{i=1}^{\infty}$ in $(C_0(Q))^*$ such that $a_n(\{y_i\}_{i=1}^{\infty})$ is not attained. Therefore by Lemma 0.1, the set $K_n(C_0(Q), c)$ is not proximinal in $K(C_0(Q), c)$. ∎

3.8. THEOREM. *Let Q and S be two locally compact Hausdorff spaces, and assume that S contains Q_0. Then for any positive integer $n \geqslant 1$, the set $K_n(C_0(Q), C_0(S))$ is proximinal in $K(C_0(Q), C_0(S))$ iff Q is finite.*

Proof. Assume that Q is infinite. By Lemma 3.7, the set $K_n(C_0(Q), c)$ is not proximinal in $K(C_0(Q), c)$, thus by Lemma 3.4, the set $K_n(C_0(Q), C_0(S))$ is not proximinal in $K(C_0(Q), C_0(S))$.

Second, assume that Q is finite. Then there is a positive integer $m \geqslant 1$, such that $C_0(Q) = l_m^{\infty}$. By Brown [1] the metric projection from $(l_m^{\infty})^* = l_m^1$ onto any of its subspaces has a continuous selection. Thus by Deutsch *et al.* [2] the set $K_n(C_0(Q), C_0(S))$ is proximinal in $K(C_0(Q), C_0(S))$. ∎

Note. Theorem 3.8 is not generally true if S fails to contain Q_0. By Deutsch *et al.* [2], the set $K_n(X, c_0)$ is proximinal in $K(X, c_0)$ for any normed linear space X, and the set $K_n(X, l_\infty)$ is proximinal in $L(X, l_\infty)$ for any normed linear space X.

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